MM2.2: Interpolation

Introduce the basic terms:

– *interpolation polynomial*;
– *Lagrange interpolation polynomial*;
– *Newton interpolation polynomial*;
– *interpolation points (nodes)*;
– *linear interpolation*;
– *piecewise linear interpolation*;
– *deviation*;
– *interpolation error*.

We shall be primarly concerned with the **interpolation** of a function of one variable: given a function $f(x)$ one chooses a function $F(x)$ from among a certain class of functions (frequently, but not always, the class of polynomials) such that $F(x)$ agrees (coincides) with $f(x)$ at certain values of x. These values of x are often referred to as interpolation points, or nodes, $x = x_k$; $k = 0, 1, \ldots$.

The actual interpolation always proceeds as follows: the function $f(x)$ to be interpolated is replaced by a function $F(x)$ which

a) deviates as little as possible from $f(x)$;
b) can be easily evaluated.

Assume that it is given a function $f(x)$ defined in an interval $a \leq x \leq b$ and its values $f(x_i) = f_i$ (ordinates) at $n+1$ different nodes $x_0, x_1, \ldots, x_n$ lying on $[a, b]$ are known. We seek to determine a polynomial $P_n(x)$ of the degree n which coincides with the given values of $f(x)$ at the interpolation points:

$$P_n(x_i) = f_i \quad (i = 0, 1, \ldots n). \tag{1}$$

The possibility to solve this problem is based on the following theorem:

there is exactly one polynomial $P_n(x)$ of degree less or equal n which satisfies the conditions (1).

For $n = 1$ when $P_1(x) = Ax + B$ is a linear function, this statement is proved below. This simplest case corresponds to the **linear interpolation** by the straight line through two points (x_0, f_0) and (x_1, f_1).

Given a function $f(x)$ defined in an interval $a \leq x \leq b$ we seek to determine a linear function $F(x)$ such that

$$f(a) = F(a), \quad f(b) = F(b). \tag{2}$$

Since $F(x)$ has the form

$$F(x) = Ax + B \tag{3}$$

for some constants A and B we have

$$F(a) = Aa + B, \qquad F(b) = Ab + B. \tag{4}$$

Solving for A and B we get

$$A = \frac{f(b) - f(a)}{b - a}, \qquad B = \frac{bf(a) - af(b)}{b - a} \tag{5}$$

and, by (2),

$$F(x) = f(a) + \frac{x-a}{b-a}(f(b) - f(a)). \tag{6}$$

One can verify directly that for each x the point $(x, F(x))$ lies on the line joining $(a, f(a))$ and $(b, f(b))$ (see Fig. 1).

We can rewrite (6) in the form

$$F(x) = w_0(x)f(a) + w_1(x)f(b) \tag{7}$$

where the "weights"

$$w_0(x) = \frac{b-x}{b-a}, \qquad w_1(x) = \frac{x-a}{b-a}. \tag{8}$$

If x lies in the interval $a \leq x \leq b$, then the weights are nonnegative and

$$w_0(x) + w_1(x) = 1, \tag{9}$$

hence, in this interval

$$0 \leq w_i(x) \leq 1, \qquad i = 0, 1. \tag{10}$$

Evidently, if $f(x)$ is a linear function, then interpolation process is exact and the function $F(x)$ from (5) coincides with $f(x)$: the bold curve and the dot line on Fig. 1 between the points $(a, f(a))$, $(b, f(b))$ coincide.

The **interpolation error**

$$\epsilon(x) = f(x) - F(x) \tag{11}$$

shows the deviation between interpolating and interpolated functions at the given point $x \in [a, b]$. Of course,

$$\epsilon(a) = \epsilon(b) = 0$$

and

$$\epsilon(x) \equiv 0, \ x \in [a, b]$$

if $f(x)$ is a linear function.

In the previous example we had only two interpolation points, $x = x_0 = a$, $x = x_1 = b$. Now let us consider the case of linear (piecewise linear) interpolation with the use of several interpolation points spaced at equal intervals:

$$x_k = a + kh; \ k = 0, 1, 2, \ldots, M, \ \ h = \frac{b-a}{M}; \ M = 2, 3, \ldots. \tag{12}$$

For a given integer M we construct the interpolating function $F(M; x)$ which is piecewise linear and which agrees with $f(x)$ at the $M + 1$ interpolation points (see Fig. 2).

In each subinterval $[x_k, \ x_{k+1}]$ we determine $F(M; x)$ by linear interpolation using the formula (5). Thus we have for $x \in [x_k, x_{k+1}]$

$$F(M; x) = f(x_k) + \frac{x - x_k}{x_{k+1} - x_k}(f(x_{k+1}) - f(x_k)). \tag{13}$$

If we denote $f_k = f(x_k)$ and use the **first forward difference** $\Delta f_k = f_{k+1} - f_k$ we obtain

$$F(M;x) = f_k + (x - x_k)\frac{\Delta f_k}{h}, \quad x \in [x_k, x_{k+1}]. \tag{14}$$

One can calculate interpolation error by the formula (11) in each subinterval and, obviously,

$$\epsilon(x_k) = 0, \;\; k = 0, 1, 2, \ldots M. \tag{15}$$

EXAMPLE 1 **Linear Lagrange interpolation**

Compute ln 9.2 from ln 9.0 = 2.1972 and ln 9.5 = 2.2513 by the linear Lagrange interpolation and determine the error from $a = \ln 9.2 = 2.2192$ (4D).

Solution. Given (x_0, f_0) and (x_1, f_1) we set

$$L_0(x) = \frac{x - x_1}{x_0 - x_1}, \quad L_1(x) = \frac{x - x_0}{x_1 - x_0},$$

which gives the Lagrange polynomial

$$p_1(x) = L_0(x)f_0 + L_1(x)f_1 = \frac{x - x_1}{x_0 - x_1}f_0 + \frac{x - x_0}{x_1 - x_0}f_1.$$

In the case under consideration, $x_0 = 9.0$, $x_1 = 9.5$, $f_0 = 2.1972$, and $f_1 = 2.2513$. Calculate

$$L_0(9.2) = \frac{9.2 - 9.5}{9.0 - 9.5} = 0.6, \quad L_1(9.2) = \frac{9.2 - 9.0}{9.5 - 9.0} = 0.4,$$

and get the answer

$$\ln 9.2 \approx \tilde{a} = p_1(9.2) = L_0(9.2)f_0 + L_1(9.2)f_1 = 0.6 \cdot 2.1972 + 0.4 \cdot 2.2513 = 2.2188.$$

The error is $\epsilon = a - \tilde{a} = 2.2192 - 2.2188 = 0.0004$.

Quadratic interpolation corresponds to the interpolation by polynomials of degree $n = 2$ when we have three (different) nodes x_0, x_1, x_2. The Lagrange polynomials of degree 2 have the form

$$\begin{aligned} l_0(x) &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}; \\ l_1(x) &= \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}; \\ l_2(x) &= \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}. \end{aligned} \tag{16}$$

Now let us form the sum

$$P_2(x) = f_0 l_0(x) + f_1 l_1(x) + f_2 l_2(x). \tag{17}$$

The following statements hold:
1) $P_2(x)$ again is a polynomial of degree 2;
2) $P_2(x_j) = f_j$, because among all terms, only $l_j(x_j) \neq 0$;

3) if there is another polynomial $Q_2(x)$ of degree 2 such that $Q_2(x_j) = f_j$, then $R_2(x) = P_2(x) - Q_2(x)$ is also a polynomial of degree 2, which vanishes at the 3 different points x_0, x_1, x_2, or, in other words, the quadratic equation $R_2(x) = 0$ has three different roots; therefore, necessarily, $R_2(x) \equiv 0$.

Hence (17), or, what is the same, the **Lagrange interpolation formula** yields the uniquely determined interpolation polynomial of degree 2 corresponding to the given interpolation points and ordinates.

Example. For $x_0 = 1,\ x_1 = 2,\ x_2 = 4$ the Lagrange polynomials (16) are:

$$l_0(x) = \frac{x-2}{1-2}\frac{x-4}{1-4} = \frac{1}{3}(x^2 - 6x + 8);$$
$$l_1(x) = \frac{x-1}{2-1}\frac{x-4}{2-4} = -\frac{1}{2}(x^2 - 5x + 4);$$
$$l_2(x) = \frac{x-1}{4-1}\frac{x-2}{4-2} = \frac{1}{6}(x^2 - 3x + 2).$$

Therefore,

$$P_2(x) = \frac{f_0}{3}(x^2 - 6x + 8) - \frac{f_1}{2}(x^2 - 5x + 4) + \frac{f_2}{6}(x^2 - 3x + 2). \tag{18}$$

If to take, for example, $f_0 = f_2 = 1,\ f_0 = 0$, then this polynomial has the form

$$P_2(x) = \frac{1}{2}(x^2 - 5x + 6) = \frac{1}{2}(x-2)(x-3)$$

and it coincides with the given ordinates at interpolation points (see Fig. 3 where the curve for $P_2(x)$ on the interpolation interval $[1, 4]$ is bold):

$$P_2(1) = 1;\ P_2(2) = 0;\ P_2(4) = 1.$$

Evidently, if $f(x)$ is a quadratic function, $f(x) = ax^2 + bx + c,\ a \neq 0$, then interpolation process is exact and the function $P_2(x)$ from (17) will coincide with $f(x)$.

The **interpolation error**

$$\epsilon(x) = f(x) - P_2(x) \tag{19}$$

shows the deviation between interpolation polynomial and interpolated function at the given point $x \in [x_0, x_2]$. Of course,

$$\epsilon(x_0) = \epsilon(x_1) = \epsilon(x_2) = 0$$

and

$$r(x) \equiv 0,\ x \in [x_0, x_2]$$

if $f(x)$ is a quadratic function.

EXAMPLE 2 **Quadratic Lagrange interpolation**

Compute $\ln 9.2$ from $\ln 9.0 = 2.1972$, $\ln 9.5 = 2.2513$, and $\ln 11.0 = 2.3979$ by the quadratic Lagrange interpolation and determine the error from $a = \ln 9.2 = 2.2192$ (4D).

Solution. Given (x_0, f_0), (x_1, f_1), and (x_2, f_2) we set

$$L_0(x) = \frac{l_0(x)}{l_0(x_0)} = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)},$$

$$L_1(x) = \frac{l_1(x)}{l_1(x_1)} = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)},$$

$$L_2(x) = \frac{l_2(x)}{l_2(x_2)} = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)},$$

which gives the quadratic Lagrange polynomial

$$p_2(x) = L_0(x)f_0 + L_1(x)f_1 + L_2(x)f_2.$$

In the case under consideration, $x_0 = 9.0$, $x_1 = 9.5$, $x_2 = 11.0$ and $f_0 = 2.1972$, $f_1 = 2.2513$, $f_2 = 2.3979$. Calculate

$$L_0(x) = \frac{(x-9.5)(x-11.0)}{(9.0-9.5)(9.0-11.0)} = x^2 - 20.5x + 104.5, \quad L_0(9.2) = 0.5400;$$

$$L_1(x) = \frac{(x-9.0)(x-11.0)}{(9.5-9.0)(9.5-11.0)} = \frac{1}{0.75}(x^2 - 20x + 99), \quad L_1(9.2) = 0.4800;$$

$$L_2(x) = \frac{(x-9.0)(x-9.5)}{(11.0-9.0)(11.0-9.5)} = \frac{1}{3}(x^2 - 18.5x + 85.5), \quad L_2(9.2) = -0.0200$$

and get the answer

$$\ln 9.2 \approx p_2(9.2) = L_0(9.2)f_0 + L_1(9.2)f_1 + L_2(9.2)f_2 =$$

$$0.5400 \cdot 2.1972 + 0.4800 \cdot 2.2513 - 0.0200 \cdot 2.3979 = 2.2192,$$

which is exact to 4D.

The **Lagrange polynomials** of degree $n = 2, 3 \ldots$ are

$$\begin{aligned} l_0(x) &= w_1^0(x)w_2^0(x)\ldots w_n^0(x); \\ l_k(x) &= w_0^k(x)w_1^k(x)\ldots w_{k-1}^k(x)w_{k+1}^k \ldots w_n^k(x), \qquad k = 1, 2 \ldots, n-1; \\ l_n(x) &= w_0^n(x)w_1^n(x)\ldots w_{n-1}^n(x), \end{aligned} \tag{20}$$

where

$$w_j^k(x) = \frac{x - x_j}{x_k - x_j}; \qquad k = 0, 1, \ldots n, \; j = 0, 1, \ldots n, \; k \neq j. \tag{21}$$

Furthermore,

$$\begin{aligned} l_k(x_k) &= 1, \\ l_k(x_j) &= 0, \; j \neq k. \end{aligned} \tag{22}$$

The **general Lagrange interpolation polynomial** is

$$\begin{aligned} P_n(x) = f_0 l_0(x) + f_1 l_1(x) + \ldots + f_{n-1} l_{n-1}(x) + f_n l_n(x), \\ n = 1, 2, \ldots \end{aligned} \tag{23}$$

and it uniquely determines the interpolation polynomial of degree n corresponding to the given interpolation points and ordinates.

Error estimate is given by

$$\epsilon_n(x) = f(x) - p_n(x) = (x - x_0)(x - x_1)\dots(x - x_n)\frac{f^{n+1}(t)}{(n+1)!},$$
$$n = 1, 2, \dots, \quad t \in (x_0, x_n)$$

if $f(x)$ has a continuous $(n+1)$st derivative.

EXAMPLE 3 **Error estimate of linear interpolation**

Solution. Given (x_0, f_0) and (x_1, f_1) we set

$$L_0(x) = \frac{x - x_1}{x_0 - x_1}, \quad L_1(x) = \frac{x - x_0}{x_1 - x_0},$$

which gives the Lagrange polynomial

$$p_1(x) = L_0(x)f_0 + L_1(x)f_1 = \frac{x - x_1}{x_0 - x_1}f_0 + \frac{x - x_0}{x_1 - x_0}f_1.$$

In the case under consideration, $x_0 = 9.0$, $x_1 = 9.5$, $f_0 = 2.1972$, and $f_1 = 2.2513$. and

$$\ln 9.2 \approx \tilde{a} = p_1(9.2) = L_0(9.2)f_0 + L_1(9.2)f_1 = 0.6 \cdot 2.1972 + 0.4 \cdot 2.2513 = 2.2188.$$

The error is $\epsilon = a - \tilde{a} = 2.2192 - 2.2188 = 0.0004$.

Estimate the error according to the general formula with $n = 1$

$$\epsilon_1(x) = f(x) - p_1(x) = (x - x_0)(x - x_1)\frac{f''(t)}{2}, \quad t \in (9.0, 9.5)$$

with $f(t) = \ln t$, $f'(t) = 1/t$, $f''(t) = -1/t^2$. Hence

$$\epsilon_1(x) = (x - 9.0)(x - 9.5)\frac{(-1)}{t^2}, \quad \epsilon_1(9.2) = (0.2)(-0.3)\frac{(-1)}{2t^2} = \frac{0.03}{t^2} \ (t \in (9.0, 9.5)),$$

$$0.00033 = \frac{0.03}{9.5^2} = \min_{t\in[9.0,9.5]}\left|\frac{0.03}{t^2}\right| \le |\epsilon_1(9.2)| \le \max_{t\in[9.0,9.5]}\left|\frac{0.03}{t^2}\right| = \frac{0.03}{9.0^2} = 0.00037$$

so that $0.00033 \le |\epsilon_1(9.2)| \le 0.00037$, which disagrees with the obtained error $\epsilon = a - \tilde{a} = 0.0004$. In fact, repetition of computations with 5D instead of 4D gives

$$\ln 9.2 \approx \tilde{a} = p_1(9.2) = 0.6 \cdot 2.19722 + 0.4 \cdot 2.25129 = 2.21885.$$

with an actual error

$$\epsilon = 2.21920 - 2.21885 = 0.00035$$

which lies in between 0.00033 and 0.00037. A discrepancy between 0.0004 and 0.00035 is thus caused by the round-off-to-4D error which is not taken into acount in the general formula for the interpolation error.

Estimation by the error principle. First we calculate

$$p_1(9.2) = 2.21885$$

and then

$$p_2(9.2) = 0.54 \cdot 2.1972 + 0.48 \cdot 2.2513 - 0.02 \cdot 2.3979 = 2.21916$$

from EXAMPLE 2 but with 5D. The difference

$$p_2(9.2) - p_1(9.2) = 2.21916 - 2.21885 = 0.00031$$

is the approximate error of $p_1(9.2)$: 0.00031 is an approximation of the error 0.00035 obtained above.

The Lagrange interpolation formula (23) is very unconvenient for actual calculation. Moreover, when computing with polynomials $P_n(x)$ for varying n, the calculation of $P_n(x)$ for a particular n is of little use in calculating a value with a larger n. These problems are avoided by using another formula for $P_n(x)$, employing the **divided differences** of the data being interpolated.

Assume that it is given the grid of points $x_0, x_1, x_2, \ldots;\ x_i \neq x_j,\ i \neq j$ and corresponding values of a function $f(x):\ f_0,\ f_1,\ f_2,\ \ldots$

The **first divided differences** are defined as

$$f[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0};\ f[x_1, x_2] = \frac{f_2 - f_1}{x_2 - x_1};\ \ldots \tag{24}$$

The **second divided differences** are

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}; \tag{25}$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1};\ \ldots \tag{26}$$

and of order n

$$f[x_0, x_1, \ldots, x_n, x_{n+1}] = \frac{f[x_1, x_2, \ldots, x_{n+1}] - f[x_0, x_1, \ldots x_n]}{x_{n+1} - x_0}. \tag{27}$$

It is easy to see that the order of $x_0,\ x_1,\ x_2, \ldots, x_n$ will not make a difference in the calculation of divided difference. In other words, any permutation of the grid points does not change the value of divided difference. Indeed, for $n = 1$

$$f[x_1, x_0] = \frac{f_0 - f_1}{x_0 - x_1} = \frac{f_1 - f_0}{x_1 - x_0} = f[x_0, x_1].$$

For $n = 2$, we obtain

$$f[x_0, x_1, x_2] = \frac{f_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{f_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{f_2}{(x_2 - x_0)(x_2 - x_1)}. \tag{28}$$

If we interchange values of x_0, x_1 and x_2, then the fractions of the right-hand side will interchange their order, but the sum will remain the same.

When the grid points (nodes) are spaced at equal intervals (forming the uniform grid), the divided difference are coupled with the **forward differences** by simple formulas. Set

$x_j = x_0 + jh$, $j = 0, 1, 2, \ldots$ and assume that $f_j = f(x_0 + jh)$ are given. Then the **first forward difference** is

$$f[x_0, x_1] = f[x_0, x_0 + h] = \frac{f(x_0 + h) - f(x_0)}{x_0 + h - x_0} = \frac{f_1 - f_0}{h} = \frac{\Delta f_0}{1!h}. \tag{29}$$

For the second forward difference we have

$$f[x_0, x_1, x_2] = \frac{1}{2h}\left(\frac{\Delta f_1}{1!h} - \frac{\Delta f_0}{1!y}\right) = \frac{\Delta^2 f_0}{2!h^2} \tag{30}$$

and so on. For arbitrary $n = 1, 2, \ldots$

$$f[x_0, x_0 + h, \ldots, x_0 + nh] = \frac{\Delta^n f_0}{n!h^n}. \tag{31}$$

It is easy to calculate divided differences using the **table of divided differences**

$$\begin{array}{llllll}
x_0 & f(x_0) & & & & \\
 & & f[x_0, x_1] & & & \\
x_1 & f(x_1) & & f[x_0, x_1, x_2] & & \\
 & & f[x_1, x_2] & & f[x_0, x_1, x_2, x_3] & \\
x_2 & f(x_2) & & f[x_1, x_2, x_3] & & \\
 & & f[x_2, x_3] & & f[x_1, x_2, x_3, x_4] & \ldots \\
\vdots & & \vdots & & \ddots & \\
 & & f[x_{n-1}, x_n] & & & \\
x_n & f(x_n) & & & &
\end{array}$$

Construct the table of divided differences for the function

$$f(x) = \frac{1}{1 + x^2},$$

at the nodes $x_k = kh$, $k = 0, 1, 2, \ldots, 10$ with the step $h = 0.1$.

The values of $f_k = f(x_k)$ are found using the table of the function $\frac{1}{x}$ with (fixed number of) three decimal places. In the first column we place the values x_k, in the second, f_k, and in the third, the first divided difference $f[x_k, x_{k+1}] = \frac{\Delta f_k}{h}$, etc.:

$$\begin{array}{lllll}
0.0 & 1.000 & & & \\
 & & -0.100 & & \\
0.1 & 0.990 & & -0.900 & \\
 & & -0.280 & & 0.167 \\
0.2 & 0.962 & & -0.850 & \\
 & & -0.450 & & \\
0.3 & 0.917 & & &
\end{array}$$

Let $P_n(x)$ denote the polynomial interpolating $f(x)$ at the nodes x_i for $i = 0, 1, 2, \ldots, n$. Thus, in general, the degree of $P_n(x)$ is less or equal n and

$$P_n(x_i) = f(x_i), \quad i = 0, 1, 2, \ldots n. \tag{32}$$

Then

$$P_1(x) = f_0 + (x - x_0)f[x_0, x_1]; \tag{33}$$

$$P_2(x) = f_0 + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] \tag{34}$$

$$\vdots$$

$$P_n(x) = f_0 + (x - x_0)f[x_0, x_1] + \dots + (x - x_0)(x - x_1)(x - x_{n-1})f[x_0, x_1, \dots x_n]. \tag{35}$$

This is called the **Newton's divided difference formula** for the interpolation polynomial. Note that for $k \geq 0$

$$P_{k+1} = P_k(x) + (x - x_0)\dots(x - x_k)f[x_0, x_1, \dots x_{k+1}]. \tag{36}$$

Thus we can go from degree k to degree $k+1$ with a minimum of calculation, once the divided differences have been computed (for example, with the help of the table of finite differences).

We will consider only the proof of (33) and (34). For the first case, one can see that $P_1(x_0) = f_0$ and

$$\begin{aligned} P_1(x_1) &= f_0 + (x_1 - x_0)\frac{f(x_1) - f(x_0)}{x_1 - x_0} \\ &= f_0 + (f_1 - f_0) = f_1. \end{aligned}$$

Thus $P_1(x)$ is a required interpolation polynomial, namely, it is a linear function which satisfies the interpolation conditions (32).

For the formula (34) we have the polynomial of a degree ≤ 2

$$P_2(x) = P_1(x) + (x - x_0)(x - x_1)f[x_0, x_1, x_2] \tag{37}$$

and for x_0, x_1

$$P_2(x_i) = P_1(x_i) + 0 = f_i, \;\; i = 0, 1.$$

Also,

$$\begin{aligned} P_2(x_2) &= f_0 + (x_2 - x_0)f[x_0, x_1] + (x_2 - x_0)(x_2 - x_1)f[x_0, x_1, x_2] \\ &= f_0 + (x_2 - x_0)f[x_0, x_1] + (x_2 - x_1)(f[x_1, x_2] - f[x_0, x_1]) \\ &= f_0 + (x_1 - x_0)f[x_0, x_1] + (x_2 - x_1)f[x_1, x_2] \\ &= f_0 + (f_1 - f_0) + (f_2 - f_1) = f_2. \end{aligned}$$

By the uniqueness of interpolation polynomial this is the quadratic interpolation polynomial for the function $f(x)$ at x_0, x_1, x_2.

In the general case the formula for the interpolation error may be represented as follows

$$f(x) = P_n(x) + \epsilon_n(x), \;\; x \in [x_0, x_n] \tag{38}$$

where $x_0, x_1, \dots, x_n$ are (different) interpolation points, $P_n(x)$ is interpolation polynomial constructed by any of the formulas verified above and $\epsilon_n(x)$ is the interpolation error.

EXAMPLE 4 **Newton's divided difference interpolation formula**

Compute $f(9.2)$ from the given values.

$$\begin{array}{llccc}
8.0 & 2.079442 & & & \\
 & & 0.117783 & & \\
9.0 & 2.197225 & & -0.006433 & \\
 & & 0.108134 & & 0.000411 \\
9.5 & 2.251292 & & -0.005200 & \\
 & & 0.097735 & & \\
11.0 & 2.397895 & & &
\end{array}$$

We have

$$f(x) \approx p_3(x) = 2.079442 + 0.117783(x - 8.0) -$$
$$0.006433(x - 8.0)(x - 9.0) + 0.000411(x - 8.0)(x - 9.0)(x - 9.5).$$

At $x = 9.2$,

$$f(9.2) \approx 2.079442 + 0.141340 - 0.001544 - 0.000030 = 2.219208.$$

We can see how the accuracy increases from term to term:

$$p_1(9.2) = 2.220782, \quad p_2(9.2) = 2.219238, \quad p_3(9.2) = 2.219208.$$

Note that interpolation makes sense only in the closed interval between the first (minimal) and the last (maximal) interpolation points.

The Newton's interpolation formula (35) becomes especially simple when interpolation points x_0, $x_0 + h$, $x_0 + 2h$, ... are spaced at equal intervals. In this case one can rewrite (38) using the formulas for divided differences and introducing the variable

$$r = \frac{x - x_0}{h} \tag{39}$$

such that

$$x = x_0 + rh,\ x - x_0 = rh,\ (x - x_0)(x - x_0 - h) = r(r-1)h^2, \ldots \tag{40}$$

Substituting this new variable into (13) and (16) we obtain the **Newton's forward difference interpolation formula**

$$f(x) \approx P_n(x) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!}\Delta^2 f_0 + \ldots + \frac{r(r-1)\ldots(r-n+1)}{n!}\Delta^n f_0. \tag{41}$$

Error estimate is given by

$$\epsilon_n(x) = f(x) - p_n(x) = \frac{h^{n+1}}{(n+1)!} r(r-1)\ldots(r-n) f^{n+1}(t),$$
$$n = 1, 2, \ldots, \quad t \in (x_0, x_n)$$

if $f(x)$ has a continuous $(n+1)$st derivative.

EXAMPLE 5 **Newton's forward difference formula. Error estimation**

Compute cosh(0.56) from the given values and estimate the error.

Solution. Construct the table of forward differences

$$
\begin{array}{llllll}
0.5 & 1.127626 & & & \\
 & & 0.057839 & & \\
0.6 & 1.185645 & & 0.011865 & \\
 & & 0.069704 & & 0.000697 \\
0.7 & 1.255169 & & 0.012562 & \\
 & & 0.082266 & & \\
0.8 & 1.337435 & & &
\end{array}
$$

In (39), we have

$$x = 0.56, \quad x_0 = 0.50, \quad h = 0.1, \quad r = \frac{x - x_0}{h} = \frac{0.56 - 0.50}{0.1} = 0.6$$

and

$$\cosh(0.56) \approx p_3(0.56) = 1.127626 + 0.6 \cdot 0.057839 + \frac{0.6(-0.4)}{2} \cdot 0.011865 + \frac{0.6(-0.4)(-1.4)}{6} \cdot 0.000697 =$$

$$1.127626 + 0.034703 - 0.001424 + 0.000039 = 1.160944.$$

Error estimate. We have $f(t) = \cosh(t)$ with $f^{(4)}(t) = \cosh^{(4)}(t) = \cosh(t)$, $n = 3$, $h = 0.1$, and $r = 0.6$, so that

$$\epsilon_3(0.56) = \cosh(0.56) - p_3(0.56) = \frac{(0.1)^4}{(4)!} 0.6(0.6-1)(0.6-2)(0.6-3)\cosh^{(4)}(t) = A\cosh(t),$$
$$t \in (0.5, 0.8)$$

where $A = -0.0000036$ and

$$A \cosh 0.8 \le \epsilon_3(0.56) \le A \cosh 0.5$$

so that

$$p_3(0.56) + A\cosh 0.8 \le \cosh(0.56) \le p_3(0.56) + A \cosh 0.5.$$

Numerical values

$$1.160939 \le \cosh(0.56) \le 1.160941$$

PROBLEM 17.3.1

Compute $\ln 9.3$ from $\ln 9.0 = 2.1972$ and $\ln 9.5 = 2.2513$ by the linear Lagrange interpolation and determine the error from $a = \ln 9.3 = 2.2300$ (exact to 4D).

Solution. Given (x_0, f_0) and (x_1, f_1) we set

$$L_0(x) = \frac{x - x_1}{x_0 - x_1}, \quad L_1(x) = \frac{x - x_0}{x_1 - x_0},$$

which gives the Lagrange polynomial

$$p_1(x) = L_0(x)f_0 + L_1(x)f_1 = \frac{x - x_1}{x_0 - x_1}f_0 + \frac{x - x_0}{x_1 - x_0}f_1.$$

In the case under consideration, $x_0 = 9.0$, $x_1 = 9.5$, $f_0 = 2.1972$, and $f_1 = 2.2513$.

$$L_0(x) = \frac{x - 9.5}{(-0.5)} = 2(9.5 - x) = 19 - 2x, \quad L_1(x) = \frac{x - 9.0}{0.5} = 2(x - 9) = 2x - 18.$$

The Lagrange polynomial is

$$p_1(x) = L_0(x)f_0 + L_1(x)f_1 =$$

$$(19-2x)2.1972+(2x-18)2.2513 = 2x(2.2513-2.1972)+19\cdot 2.1972-18\cdot 2.2513 = 0.1082x+1.2234.$$

Now calculate

$$L_0(9.3) = \frac{9.3 - 9.5}{9.0 - 9.5} = 0.4, \quad L_1(9.3) = \frac{9.3 - 9.0}{9.5 - 9.0} = 0.6,$$

and get the answer

$$\ln 9.3 \approx \tilde{a} = p_1(9.3) = L_0(9.3)f_0 + L_1(9.3)f_1 = 0.4 \cdot 2.1972 + 0.6 \cdot 2.2513 = 2.2297.$$

The error is $\epsilon = a - \tilde{a} = 2.2300 - 2.2297 = 0.0003$.

PROBLEM 17.3.2

Estimate the error of calculating $\ln 9.3$ from $\ln 9.0 = 2.1972$ and $\ln 9.5 = 2.2513$ by the linear Lagrange interpolation ($\ln 9.3 = 2.2300$ exact to 4D).

Solution. We estimate the error according to the general formula with $n = 1$

$$\epsilon_1(x) = f(x) - p_1(x) = (x - x_0)(x - x_1)\frac{f''(t)}{2}, \quad t \in (9.0, 9.5)$$

with $f(t) = \ln t$, $f'(t) = 1/t$, $f''(t) = -1/t^2$. Hence

$$\epsilon_1(x) = (x - 9.0)(x - 9.5)\frac{(-1)}{t^2}, \quad \epsilon_1(9.3) = (0.3)(-0.2)\frac{(-1)}{2t^2} = \frac{0.03}{t^2} \ (t \in (9.0, 9.5)),$$

$$0.00033 = \frac{0.03}{9.5^2} = \min_{t\in[9.0,9.5]} \left|\frac{0.03}{t^2}\right| \le |\epsilon_1(9.3)| \le \max_{t\in[9.0,9.5]} \left|\frac{0.03}{t^2}\right| = \frac{0.03}{9.0^2} = 0.00037$$

so that $0.00033 \le |\epsilon_1(9.3)| \le 0.00037$, which disagrees with the obtained error $\epsilon = a - \tilde{a} = 0.0003$ because in the 4D computations we cannot round-off the last digit 3. In fact, repetition of computations with 5D instead of 4D gives

$$\ln 9.3 \approx \tilde{a} = p_1(9.3) = 0.4 \cdot 2.19722 + 0.6 \cdot 2.25129 = 0.87889 + 1.35077 = 2.22966$$

with an actual error

$$\epsilon = 2.23001 - 2.22966 = 0.00035$$

which lies between 0.00033 and 0.00037. A discrepancy between 0.0003 and 0.00035 is thus caused by the round-off-to-4D error which is not taken into acount in the general formula for the interpolation error.

PROBLEM 17.3.3

Compute $e^{-0.25}$ and $e^{-0.75}$ by linear interpolation with $x_0 = 0$, $x_1 = 0.5$ and $x_0 = 0.5$, $x_1 = 1$. Then find $p_2(x)$ interpolating e^{-x} with $x_0 = 0$, $x_1 = 0.5$, and $x_2 = 1$ and from it $e^{-0.25}$ and $e^{-0.75}$. Compare the errors of these linear and quadratic interpolation.

Solution. Given (x_0, f_0) and (x_1, f_1) we set

$$L_0(x) = \frac{x - x_1}{x_0 - x_1}, \quad L_1(x) = \frac{x - x_0}{x_1 - x_0},$$

which gives the Lagrange polynomial

$$p_1(x) = L_0(x)f_0 + L_1(x)f_1 = \frac{x - x_1}{x_0 - x_1}f_0 + \frac{x - x_0}{x_1 - x_0}f_1.$$

In the case of linear interpolation, we will interpolate e^x and take first the nodes $x_0 = -0.5$ and $x_1 = 0$ and $f_0 = e^{-0.5} = 0.6065$ and $f_1 = e^0 = 1.0000$.

$$L_0(x) = \frac{x - 0}{(-0.5)} = -2x, \quad L_1(x) = \frac{x + 0.5}{(0.5)} = 2(x + 0.5) = 2x + 1.$$

The Lagrange polynomial is

$$p_1(x) = L_0(x)f_0 + L_1(x)f_1 =$$

$$-2x \cdot 0.6065 + (2x + 1)1.0000 = 2x(1.0000 - 0.6065) + 1 = 2 \cdot 0.3935x + 1.$$

The answer

$$e^{-0.25} \approx p_1(-0.25) = -0.25 \cdot 2 \cdot 0.3935 + 1 = 1 - 0.1967 = 0.8033.$$

The error is $\epsilon = e^{-0.25} - p_1(-0.25) = 0.7788 - 0.8033 = -0.0245$.

Now take the nodes $x_0 = -1$ and $x_1 = -0.5$ and $f_0 = e^{-1} = 0.3679$ and $f_1 = e^{-0.5} = 0.6065$.

$$L_0(x) = \frac{x + 0.5}{(-0.5)} = -2(x + 0.5) = -2x - 1, \quad L_1(x) = \frac{x + 1}{(0.5)} = 2(x + 1).$$

The Lagrange polynomial is

$$p_1(x) = L_0(x)f_0 + L_1(x)f_1 =$$

$$(-2x-1)\cdot 0.3679 + (2x+2)0.6065 = 2x(0.6065 - 0.3679) - 0.3679 + 2\cdot 0.6065 = 2\cdot 0.2386x + 0.8451.$$

The answer

$$e^{-0.75} \approx p_1(-0.75) = -0.75 \cdot 2 \cdot 0.2386 + 0.8451 = -0.3579 + 0.8451 = 0.4872.$$

The error is $\epsilon = e^{-0.75} - p_1(-0.75) = 0.4724 - 0.4872 = -0.0148$.

In the case of quadratic interpolation, we will again interpolate e^x and take the nodes $x_0 = -1$, $x_1 = -0.5$, and $x_2 = 0$ and $f_0 = e^{-1} = 0.3679$, $f_1 = e^{-0.5} = 0.6065$ and $f_2 = 1.0000$.

Given (x_0, f_0), (x_1, f_1), and (x_2, f_2) we set

$$L_0(x) = \frac{l_0(x)}{l_0(x_0)} = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)},$$

$$L_1(x) = \frac{l_1(x)}{l_1(x_1)} = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)},$$

$$L_2(x) = \frac{l_2(x)}{l_2(x_2)} = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)},$$

which gives the quadratic Lagrange polynomial

$$p_2(x) = L_0(x)f_0 + L_1(x)f_1 + L_2(x)f_2.$$

In the case under consideration, calculate

$$L_0(x) = \frac{(x+0.5)(x)}{(-0.5)(-1)} = 2x(x+0.5);$$

$$L_0(-0.25) = -0.5 \cdot 0.25 = -0.125, \quad L_0(-0.75) = (-1.5)\cdot(-0.25) = 0.375.$$

$$L_1(x) = \frac{(x+1)(x)}{(0.5)(-0.5)} = -4x(x+1);$$

$$L_1(-0.25) = 1 \cdot 0.75 = 0.75, \quad L_1(-0.75) = 3 \cdot 0.25 = 0.75.$$

$$L_2(x) = \frac{(x+1)(x+0.5)}{(1)(0.5)} = 2(x+0.5)(x+1);$$

$$L_2(-0.25) = 0.5 \cdot 0.75 = 0.375, \quad L_2(-0.75) = (-0.5)\cdot 0.25 = -0.125.$$

The answers are as follows:

$$e^{-0.25} \approx p_2(-0.25) = L_0(-0.25)f_0 + L_1(-0.25)f_1 + L_2(-0.25)f_2 =$$

$$-0.1250 \cdot 0.3679 + 0.7500 \cdot 0.6065 + 0.3750 \cdot 1.0000 = -0.0460 + 0.4549 + 0.3750 = 0.7839.$$

The error is $\epsilon = e^{-0.25} - p_2(-0.25) = 0.7788 - 0.7839 = -0.0051.$.

$$e^{-0.75} \approx p_2(-0.75) = L_0(-0.75)f_0 + L_1(-0.75)f_1 + L_2(-0.75)f_2 =$$

$$0.3750 \cdot 0.3679 + 0.7500 \cdot 0.6065 - 0.1250 \cdot 1.0000 = 0.1380 + 0.4549 - 0.1250 = 0.4679.$$

The error is $\epsilon = e^{-0.75} - p_2(-0.75) = 0.4724 - 0.4679 = 0.0045.$.

The quadratic Lagrange polynomial is

$$p_2(x) = L_0(x)f_0 + L_1(x)f_1 + L_2(x)f_2 =$$

$$2x(x+0.5)\cdot 0.3679 - 4x(x+1)\cdot 0.6065 + 2(x+0.5)(x+1)\cdot 1.0000 = 0.3095x^2 - 0.9418x + 1.$$

PROBLEM 17.3.5 (**quadratic interpolation**)

Calculate the Lagrange polynomial $p_2(x)$ for 4-D values of the Gamma-function $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$, $\Gamma(1.00) = 1.0000$, $\Gamma(1.02) = 0.9888$, and $\Gamma(1.04) = 0.9784$, and from it the approximation of $\Gamma(1.01)$ and $\Gamma(1.03)$.

Solution. We will interpolate $\Gamma(x)$ taking the nodes $x_0 = 1.00$, $x_1 = 1.02$, and $x_2 = 1.04$ and $f_0 = 1.0000$, $f_1 = 0.9888$, and $f_2 = 0.9784$.

Given (x_0, f_0), (x_1, f_1), and (x_2, f_2) we set

$$L_0(x) = \frac{l_0(x)}{l_0(x_0)} = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)},$$

$$L_1(x) = \frac{l_1(x)}{l_1(x_1)} = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)},$$

$$L_2(x) = \frac{l_2(x)}{l_2(x_2)} = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)},$$

which gives the quadratic Lagrange polynomial

$$p_2(x) = L_0(x)f_0 + L_1(x)f_1 + L_2(x)f_2.$$

In the case under consideration, calculate

$$L_0(x) = \frac{(x-1.02)(x-1.04)}{(-0.02)(-0.04)} = 1250(x-1.02)(x-1.04);$$

$$L_0(1.01) = \frac{(-0.01)(-0.03)}{(-0.02)(-0.04)} = \frac{3}{8} = 0.375, \quad L_0(1.03) = \frac{(0.01)(-0.01)}{(-0.02)(-0.04)} = -\frac{1}{8} = -0.125.$$

$$L_1(x) = \frac{(x-1)(x-1.04)}{(0.02)(-0.02)} = -2500(x-1)(x-1.04);$$

$$L_1(1.01) = \frac{(0.01)(-0.03)}{(0.02)(-0.02)} = \frac{3}{4} = 0.75, \quad L_1(1.03) = \frac{(0.03)(-0.01)}{(0.02)(-0.02)} = \frac{3}{4} = 0.75.$$

$$L_2(x) = \frac{(x-1)(x-1.02)}{(0.04)(0.02)} = 1250(x-1)(x-1.02);$$

$$L_2(1.01) = \frac{(0.01)(-0.01)}{(0.04)(0.02)} = -\frac{1}{8} = -0.125, \quad L_1(1.03) = \frac{(0.03)(0.01)}{(0.04)(0.02)} = \frac{3}{8} = 0.375.$$

The answers are as follows:

$$\Gamma(1.01) \approx p_2(1.01) = L_0(1.01)f_0 + L_1(1.01)f_1 + L_2(1.01)f_2 =$$

$$0.3750 \cdot 1.0000 + 0.7500 \cdot 0.9888 - 0.1250 \cdot 0.9784 = 0.3750 + 0.7416 - 0.1223 = 0.9943.$$

The error is $\epsilon = \Gamma(1.01) - p_2(1.01) = 0.9943 - 0.9943 = 0.0000$. The result is exact to 4D.

$$\Gamma(1.03) \approx p_2(1.03) = L_0(1.03)f_0 + L_1(1.03)f_1 + L_2(1.03)f_2 =$$

$$-0.1250 \cdot 1.0000 + 0.7500 \cdot 0.9888 + 0.3750 \cdot 0.9784 = -0.1250 + 0.7416 + 0.3669 = 0.9835.$$

The error is $\epsilon = \Gamma(1.03) - p_2(1.03) = 0.9835 - 0.9835 = 0.0000$. The result is exact to 4D.

The quadratic Lagrange polynomial is

$$p_2(x) = L_0(x)f_0 + L_1(x)f_1 + L_2(x)f_2 =$$

$$1250(x-1.02)(x-1.04)\cdot 1.0000 - 2500(x-1)(x-1.04)\cdot 0.9888 + 1250(x-1)(x-1.02)\cdot 0.9784 =$$

$$= x^2(1250(\cdot 1.9784 - 2\cdot 0.9888)) + \ldots = x^2(1250((2-0.0216) - 2(1-\cdot 0.0112))) + \ldots =$$

$$x^2(1250(-0.0216 + 2\cdot 0.0112)) + \ldots = x^2(1250\cdot 0.0008) + \ldots = x^2\cdot 1.000 + \ldots = x^2 - 2.580x + 2.580.$$

PROBLEM 17.3.11 (**Newton's forward difference formula**)

Compute $\Gamma(1.01)$, $\Gamma(1.03)$, and $\Gamma(1.05)$ by Newton's forward difference formula.

Solution. Construct the table of forward differences

$$\begin{array}{cccc} 1.00 & 1.0000 & & \\ & & 0.0112 & \\ 1.02 & 0.9888 & & 0.0008 \\ & & 0.0104 & \\ 1.04 & 0.9784 & & \end{array}$$

From Newton's forward difference formula, we have

$$x = 1.01,\ 1.03,\ 1.05; \quad x_0 = 1.00, \quad h = 0.02, \quad r = \frac{x-1}{h} = 50(x-1);$$

$$p_2(x) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2}\Delta^2 f_0 = 1.000 - 0.0112r + 0.0008\frac{r(r-1)}{2} = x^2 - 2.580x + 2.580,$$

which coincides with the quadratic interpolation polynomial derived above. Therefore, we can perform direct calculations to obtain

$$\Gamma(1.01) \approx p_2(1.01) = 0.9943, \quad \Gamma(1.03) \approx p_2(1.03) = 0.9835, \quad \Gamma(1.05) \approx p_2(1.05) = 0.9735.$$

PROBLEM 17.3.13 (**Lower degree**)

What is the degree of the interpolation polynomial for the data (1,5), (2,18), (3,37), (4,62), (5,93)?

Solution. We find the polynomial proceeding from an assumption that it has the lowest possible degree $n = 2$ and has the form $p_2(x) = ax^2 + bx + c$. In fact, it is easy to check that the required polynomial is not a linear function, $Ax + B$, because

$$\begin{aligned} A + B &= 5 \\ 2A + B &= 18 \end{aligned}$$

yields $A = 13$ and $B = -8$, and

$$\begin{aligned} 3A + B &= 37 \\ 4A + B &= 62 \end{aligned}$$

yields $A = 25$ and $B = -38$.

For the determination of the coefficients a, b, c we have the system of three linear algebraic equations

$$\begin{aligned} a + b + c &= 5, \\ 4a + 2b + c &= 18 \\ 9a + 3b + c &= 37 \end{aligned}$$

Subtracting the first equation from the second and from the third, we obtain

$$\begin{aligned} 3a + b &= 13 \\ 8a + 2b &= 32 \end{aligned}$$

or

$$\begin{aligned} 6a + 2b &= 26 \\ 8a + 2b &= 32 \end{aligned}$$

which yields $2a = 6$, $a = 3$, $b = 13 - 3a = 4$, and $c = 5 - b - a = -2$.

Thus, the desired polynomial of degree $n = 2$ is $p_2(x) = 3x^2 + 4x - 2$.

It is easy to see that $p_2(1) = 5$, $p_2(2) = 18$, $p_2(3) = 37$, $p_2(4) = 62$, and $p_2(5) = 93$.